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THE NON-ORIENTABLE GENUS OF SOME METACYCLIC GROUPS

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We describe non-orientable, octagonal embeddings for certain 4-valent, bipartite Cayley graphs of finite metacyclic groups, and give a class of examples for which this embedding realizes the non-orientable genus of the group. This yields a construction of Cayley graphs for which $2\gamma - \tilde{\gamma}$ is arbitrarily large, where γ and $\tilde{\gamma}$ are the orientable genus and the non-orientable genus of the Cayley graph.

1. Introduction

It is well-known that if a regular d-valent graph on v vertices and of girth g admits an embedding into an orientable surface of genus γ, then

$$\gamma \geq \left\lceil 1 - \frac{v}{2} + \frac{vd}{4}\left(1 - \frac{2}{g}\right)\right\rceil . \tag{1.1}$$

In the case of a 2-cell embedding (i.e., in the case where every region of the embedding is homeomorphic to a disk), the inequality can easily be derived from the Euler Formula, $v - e + f = 2 - 2\gamma$ (where e is the number of edges and f is the number of 2-cells of the embedding): the Handshaking Lemma asserts $2e = dv$, and applying the Handshaking Lemma to the dual graph yields $2e \geq gf$; incorporating these observations into the Euler Formula and using the fact that γ is an integer results in precisely (1.1). For an argument extending this inequality to the case where not every region is homeomorphic to a disk, see [12].

A similar result is true for an embedding into a non-orientable surface of non-orientable genus $\tilde{\gamma}$ [8, Theorem 2b]. In this case, we use the non-orientable version of the Euler Formula, $v - e + f = 2 - \tilde{\gamma}$, and the conclusion is

$$\tilde{\gamma} \geq \left\lceil 2 - v + \frac{vd}{2}\left(1 - \frac{2}{g}\right)\right\rceil . \tag{1.2}$$

The *genus* of a graph is the minimum of the genera of the orientable surfaces on which the graph can be embedded; the *genus* of a finite group [11], [7], [3] is

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the minimum of the genera of the Cayley graphs of the group. (To find the genus of a finite group, it suffices to consider only irredundant Cayley graphs.) The *non-orientable genus* of a graph or of a group is defined similarly. Most groups whose genus (or non-orientable genus) is known have an irredundant Cayley graph that yields equality in (1.1) (or in (1.2), respectively) and, at the same time, achieves the minimum (among all the irredundant Cayley graphs of the group) for the right-hand side of this inequality. Such groups are said to admit *best* (orientable or non-orientable) embeddings. Groups that do not admit best embeddings are much more difficult to deal with. Here we describe non-orientable 2-cell embeddings for certain Cayley graphs of metacyclic groups, and give a class of groups for which this embedding is a best non-orientable embedding.

For a Cayley graph of orientable genus γ and non-orientable genus $\tilde{\gamma}$, the quantity $2\gamma - \tilde{\gamma}$ is a measure of the difference between the minimal orientable embeddings and the minimal non-orientable embeddings of the Cayley graph [8]. (This quantity compares the Euler characteristic $2 - 2\gamma$ of an orientable surface with the Euler characteristic $2 - \tilde{\gamma}$ of a non-orientable surface.) Every graph satisfies $2\gamma - \tilde{\gamma} \geq -1$ [9, Theorem 7]. There are examples of graphs of non-orientable genus one and arbitrarily large orientable genus [1]. For groups, Brin, Rauschenberg, and Squier [2] have shown that $2\gamma - \tilde{\gamma} = 3$ when Γ is the nonabelian, metacyclic group of order 27. In this paper, we construct a family of Cayley graphs for which $2\gamma - \tilde{\gamma}$ is arbitrarily large; these are the first known Cayley graphs for which $2\gamma - \tilde{\gamma} > 3$. It would be interesting to construct a family of groups for which $2\gamma - \tilde{\gamma}$ is arbitrarily large. The groups constructed in this paper are good candidates, but we have an interesting lower bound on the orientable genus only for certain of their Cayley graphs; we do not know how to prove an interesting lower bound that holds for *all* of the Cayley graphs of the groups.

(3.1′) Proposition. *Suppose $\{x, y\}$ is an irredundant, 2-element generating set for a finite group Γ. If the corresponding Cayley graph,* $\operatorname{Cay}(\Gamma; x^{\pm1}, y^{\pm1})$, *is bipartite and 4-valent, and if the subgroup generated by x is a normal subgroup of Γ, then* $\operatorname{Cay}(\Gamma; x^{\pm1}, y^{\pm1})$ *has an octagonal, non-orientable embedding; therefore,* $\tilde{\gamma}(\Gamma) \leq 2 + |\Gamma|/2$.

(4.1′) Theorem. *Let m, n, and k be powers of 2, with $64 \leq 8k \leq m \leq nk$. Then the embedding of Proposition 3.1 is a best non-orientable embedding for the metacyclic group given by the following generators and relations:*

$$\langle x, y \mid x^m = y^n = e,\ y^{-1}xy = x^{k+1} \rangle.$$

In particular, the non-orientable genus of this group is $2 + mn/2$.

Remark. Corollary 4.2 gives some additional examples where the embedding of Proposition 3.1 is a best non-orientable embedding.

(5.2′) Theorem. *Let m, n, and k be powers of 2, with $256 \leq k^2 < m \leq nk$, and let Γ be the metacyclic group given by the following generators and relations:*

$$\langle x, y \mid x^m = y^n = e,\ y^{-1}xy = x^{k+1} \rangle.$$

Then the Cayley graph $G = \operatorname{Cay}(\Gamma; x^{\pm1}, y^{\pm1})$ *satisfies* $2\gamma(G) - \tilde{\gamma}(G) \geq n/10$.

2. Preliminaries: group theory and Cayley graphs

Definition. A group Γ is *metacyclic* if Γ has a normal subgroup N such that the subgroup N is cyclic, and the quotient group Γ/N is also cyclic.

(2.1) Remark. (see [4, §1.8, pp. 9–10]). Any finite, metacyclic group has a presentation of the form

$$\langle x, y \mid x^m = e,\ y^n = x^r,\ y^{-1}xy = x^k \rangle,$$

for some natural numbers m, n, r, k such that $k^n \equiv 1 \pmod{m}$, $kr \equiv r \pmod{m}$, and $\gcd(k, m) = 1$. Conversely, for any such natural numbers, the given presentation defines a metacyclic group of order mn.

Definition. For any subset Δ of a group Γ, we use $\langle \Delta \rangle$ to denote the subgroup of Γ generated by Δ. We say that Δ is a *generating set* for Γ if $\langle \Delta \rangle = \Gamma$, and that Δ is *symmetric* if, for every $x \in \Delta$, we have $x^{-1} \in \Delta$.

Definition. Suppose Δ is a symmetric generating set for a group Γ, and that Δ does not contain the identity element e of Γ. Then the *Cayley graph* $\mathrm{Cay}(\Gamma; \Delta)$ is a graph defined as follows. The vertices of $\mathrm{Cay}(\Gamma; \Delta)$ are the elements of Γ; for each $g \in \Gamma$ and each $x \in \Delta$, there is an edge joining g and gx.

Definition. A symmetric generating set Δ for a finite group Γ is *irredundant* if $\langle \Delta \setminus \{x, x^{-1}\} \rangle$ is a proper subgroup of Γ, for every $x \in \Delta$. We say that the Cayley graph $\mathrm{Cay}(\Gamma; \Delta)$ is *irredundant* if Δ is an irredundant symmetric generating set.

Definition. [5, p. 173]. The *Frattini subgroup* $\Phi(\Gamma)$ of a finite group Γ is the intersection of all the maximal subgroups of Γ.

One can show (see [5, Theorem 5.1.1(i)]) that an element x of Γ belongs to $\Phi(\Gamma)$ if and only if x belongs to no irredundant symmetric generating set for Γ. The following lemma is another way of saying essentially the same thing.

(2.2) Lemma. [5, Theorem 5.1.1(i)]. *Let* $\mathrm{Cay}(\Gamma; \Delta)$ *be an irredundant Cayley graph of a finite group* Γ, *and let* $\overline{\Gamma} = \Gamma/\Phi(\Gamma)$. *Then* $\mathrm{Cay}(\overline{\Gamma}; \overline{\Delta})$ *is an irredundant Cayley graph of* $\overline{\Gamma}$, *where* $\overline{\Delta}$ *is the image of* Δ *under the natural homomorphism* $\Gamma \to \overline{\Gamma}$. ∎

(2.3) Lemma. [5, Theorem 5.1.3]. *If* Γ *is a finite p-group, then* $\Phi(\Gamma) = \langle \Gamma^p, [\Gamma, \Gamma] \rangle$, *where* $\Gamma^p = \langle x^p \mid x \in \Gamma \rangle$. ∎

(2.4) Lemma. *Let* Γ *and* Λ *be finite groups, and assume* $\gcd(|\Gamma|, |\Lambda|) = 1$. *Then* $\Phi(\Gamma \times \Lambda) = \Phi(\Gamma) \times \Phi(\Lambda)$.

Proof. Since $|\Gamma|$ and $|\Lambda|$ are relatively prime, every subgroup of $\Gamma \times \Lambda$ is of the form $A \times B$, where A is a subgroup of Γ and B is a subgroup of Λ. Therefore, maximal subgroups of $\Gamma \times \Lambda$ are those of the form $M \times \Lambda$ or $\Gamma \times N$, where M is a maximal subgroup of Γ and N is a maximal subgroup of Λ. The conclusion follows. ∎

We need only one direction ($\Rightarrow$) of the following proposition, but we prove the converse because it provides an amusing characterization of 2-groups. A similar result appears in [10, Theorem 2.2].

(2.5) Proposition. *A finite group* Γ *is a 2-group if and only if every irredundant Cayley graph on* Γ *is bipartite.*

Proof. ($\Rightarrow$) Suppose $\mathrm{Cay}(\Gamma;\Delta)$ is an irredundant Cayley graph of Γ, where $|\Gamma|$ is a power of 2. Let $\overline{\Gamma} = \Gamma/\Phi(\Gamma)$. Lemma 2.3 implies that $\overline{\Gamma}$ is an elementary abelian 2-group (i.e., a group of the form $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$), and Lemma 2.2 asserts that $\mathrm{Cay}(\overline{\Gamma};\overline{\Delta})$ is an irredundant Cayley graph of Γ. We conclude that $\mathrm{Cay}(\overline{\Gamma};\overline{\Delta})$ is isomorphic to the n-cube graph $Q_n = K_2 \times \cdots \times K_2$, for some n; in particular, $\mathrm{Cay}(\overline{\Gamma};\overline{\Delta})$ is bipartite. Since $\mathrm{Cay}(\overline{\Gamma};\overline{\Delta})$ is a homomorphic image of $\mathrm{Cay}(\Gamma;\Delta)$, this implies $\mathrm{Cay}(\Gamma;\Delta)$ is also bipartite.

($\Leftarrow$) Because no irredundant Cayley graph on Γ has a cycle of odd length, we know that no irredundant symmetric generating set for Γ contains an element of odd order. Therefore, every odd-order element x of Γ belongs to $\Phi(\Gamma)$. This implies that $\Gamma/\Phi(\Gamma)$ is a 2-group. In particular, this means that $\Gamma/\Phi(\Gamma)$ is nilpotent. It follows that Γ itself is nilpotent [5, Theorem 6.1.6(ii), p. 219]. Thus we may write $\Gamma = P \times Q$, where P is a 2-group and Q has odd order. Because every odd-order element of Γ belongs to $\Phi(\Gamma)$, we must have $Q \subset \Phi(\Gamma)$. Lemma 2.4 asserts that $\Phi(\Gamma) = \Phi(P) \times \Phi(Q)$, so this implies that $Q \subset \Phi(Q)$. Therefore Q is trivial, so $\Gamma = P$ is a 2-group, as desired. ∎

(2.6) Lemma. *Let p be a prime number, let k and n be powers of p, and let $\Gamma = \mathbb{Z}_k \times \mathbb{Z}_n$. If $\langle a,b\rangle = \Gamma$, then, in any relation of length less than $\min(k,n)$, the number of occurrences of a equals the number of occurrences of a^{-1}, and the number of occurrences of b equals the number of occurrences of b^{-1}.*

Proof. If not, then $a^s b^t = 0$, where $0 < s < \min(k,n)$ (or $0 < t < \min(k,n)$). Hence $a^s \in \langle b\rangle$, so $|\Gamma : \langle b\rangle| \le s < \min(k,n)$. This implies the order of b is greater than $\max(k,n)$, which is impossible because no element of Γ has order greater than $\max(k,n)$. ∎

(2.7) Lemma. *Suppose Γ_1 and Γ_2 are finite metacylic groups, and that*

$$\gcd(|\Gamma_1|,|\Gamma_2|) = 1.$$

Then $\Gamma_1 \times \Gamma_2$ is also metacyclic.

Proof. Let N_i be a cyclic, normal subgroup of Γ_i such that Γ_i/N_i is cyclic, and let $N = N_1 \times N_2$. Then N is the direct product of two cyclic groups of relatively prime order, so N is cyclic. The quotient group $(\Gamma_1 \times \Gamma_2)/N \cong (\Gamma_1/N_1) \times (\Gamma_2/N_2)$ is also a direct product of two cyclic groups of relatively prime order; hence it is also cyclic. ∎

Remark. If m_i, n_i, r_i, k_i are the parameters for Γ_i $(i = 1,2)$, as described in Remark 2.1, then the parameters m,n,r,k for $\Gamma_1 \times \Gamma_2$ can be determined by the following conditions: $m = m_1 m_2$, $n = n_1 n_2$, $k \equiv k_i \pmod{m_i}$, $r \equiv r_1 n_2 \pmod{m_1}$ and $r \equiv r_2 n_1 \pmod{m_2}$.

3. Non-orientable embeddings of metacyclic Cayley graphs

(3.1) Proposition. *Let $G = \mathrm{Cay}(\Gamma; x^{\pm1}, y^{\pm1})$ be a 4-valent Cayley graph of a finite group Γ, and assume $\langle x\rangle \lhd \Gamma$. If G is bipartite, then G has an octagonal non-orientable embedding. In particular, $\tilde{\gamma}(\Gamma) \le 2 + |\Gamma|/2$.*

Proof. We describe the embedding in terms of Stahl's generalized embedding schemes [9]. For convenience, we note that each arc in G can naturally be labeled x,

x^{-1}, y, or y^{-1}. In terms of the arc labels, it is easy to define the rotation system P: the local rotation P_u at u is the cyclic permutation (x, y, x^{-1}, y^{-1}).

The definition of λ, the voltage map, is based on a bipartition $V(G) = A \cup B$ of G. It suffices to define λ on the arcs labeled x or y, because the value of λ on any arc is the same as its value on the reverse of the arc. The values of λ are 0 and 1. For all x-arcs, λ is 0. For y-arcs, λ is 0 on y-arcs that originate in A (and terminate in B); and λ is 1 on y-arcs that originate in B (and terminate in A).

It is easy to verify that the embedding is octagonal (each region is bounded by a walk of the form $xyx^{-1}y^{-1}x^{-1}yxy^{-1}$), and that the embedding is non-orientable (a closed walk of the form $yx^{-1}y^{-1}x^{r}$ is not λ-trivial). ∎

For interested readers, Figure 1 shows an embedded voltage graph for the embedding of Proposition 3.1. (See [7] for information on voltage graphs and on how to interpret the diagram.)

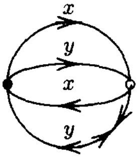

Fig.1. A 2-vertex embedded voltage graph for the embedding of Proposition 3.1. The rotation is counterclockwise at the hollow vertex and clockwise at the solid vertex

(3.2) Proposition. *Suppose Γ is a nontrivial, finite group with the following two properties:*

a) *no irredundant generating set for Γ contains an element of order 2 or an element of order 3; and*

b) *no 4-valent Cayley graph of Γ has girth less than 8.*

Then $\gamma(\Gamma) \geq 1 + |\Gamma|/4$ and $\tilde{\gamma}(\Gamma) \geq 2 + |\Gamma|/2$.

Proof. Let $G = \mathrm{Cay}(\Gamma; \Delta)$ be an irredundant Cayley graph of Γ, of girth g and valence d. Property (a) implies that Δ contains no elements of order 3, so $g \geq 4$. If $d \geq 6$, this implies $d[1 - (2/g)] \geq 3$, in which case (1.1) and (1.2) yield the desired inequalities. If $d = 4$, we conclude from Property (b) that $g \geq 8$; we again have $d[1 - (2/g)] \geq 3$.

Property (a) implies that Δ contains no elements of order 2, so d is even. Thus, the only case not covered by the previous paragraph is when $d = 2$. But this case cannot occur: a group with a 2-valent Cayley graph must either contain elements of order 2 or be cyclic (the generating set must either be of the form $\{a, b\}$ where a and b have order 2 or be of the form $\{x, x^{-1}\}$, in which case x must generate the group.) Property (a) rules out the existence of elements of order 2, and Property (b) implies that Γ is not abelian (for otherwise there would be a cycle $aba^{-1}b^{-1}$ of length 4), so Γ cannot be cyclic. ∎

(3.3) Theorem. *Let Γ be a finite group that has a bipartite, 4-valent Cayley graph $\mathrm{Cay}(\Gamma; x^{\pm 1}, y^{\pm 1})$ such that $\langle x \rangle \triangleleft \Gamma$. If Γ satisfies Properties (a) and (b) of Proposition 3.2, then $\tilde{\gamma}(\Gamma) = 2 + |\Gamma|/2$.*

Proof. Proposition 3.1 asserts one direction of inequality; Proposition 3.2 asserts the other. ∎

4. Some metacyclic groups

We show how to construct some metacyclic groups that satisfy the hypotheses of Theorem 3.3. We will first (4.1) show how to construct examples whose order is a power of 2; we will then (4.2) show how to get examples that are not 2-groups. The work relies on basic properties of the Frattini subgroup, $\Phi(\Gamma)$ (see §2).

The groups we construct are fairly large. For example, the smallest of the groups described in Theorem 4.1 has order 512. By Theorem 3.3, its non-orientable genus is 258, so its orientable genus is at least 129.

(4.1) Theorem. *Let m, n, and k be natural numbers, with $6 \leq k+3 \leq m \leq n+k$. Then the metacyclic group*

$$\Gamma = \langle x, y \mid x^{2^m} = y^{2^n} = e,\ y^{-1}xy = x^{2^k+1} \rangle$$

with its standard Cayley graph $G = \operatorname{Cay}(\Gamma; x^{\pm 1}, y^{\pm 1})$ satisfies the hypotheses of Theorem 3.3, and every 4-valent Cayley graph of Γ is bipartite.

Proof. Proposition 2.5 asserts that every irredundant Cayley graph of any 2-group is bipartite, so every 4-valent Cayley graph of Γ is bipartite. Let $\overline{\Gamma} = \Gamma/[\Gamma,\Gamma] \cong \mathbb{Z}_{2^k} \times \mathbb{Z}_{2^n}$, and let $\Gamma \to \overline{\Gamma}\colon g \mapsto \overline{g}$ be the natural homomorphism.

a) Since $|\Gamma|$ (a power of 2) is not divisible by 3, there are no elements of order 3 in Γ. Suppose $t \in \Gamma$ is an element of order 2. Now $\overline{\Gamma} \cong \mathbb{Z}_{2^k} \times \mathbb{Z}_{2^n}$, and $k, n > 1$, so every element of order ≤ 2 in $\overline{\Gamma}$ is a square; in particular, $\overline{t} \in \overline{\Gamma}^2$. Hence $t \in \langle \Gamma^2, [\Gamma,\Gamma] \rangle = \Phi(\Gamma)$, so t cannot belong to an irredundant generating set for Γ.

b) Suppose some 4-valent Cayley graph $\operatorname{Cay}(\Gamma; a^{\pm 1}, b^{\pm 1})$ has girth less than 8. Each arc of the Cayley graph can naturally be labeled a, a^{-1}, b, or b^{-1}; in any cycle of length less than $\min(2^k, 2^n)$ (in particular, in any cycle of length less than 8), Lemma 2.6 implies that the number of occurrences of arcs labeled a must equal the number of occurrences of arcs labeled a^{-1}, and the number of occurrences of arcs labeled b must equal the number of occurrences of arcs labeled b^{-1}. So a cycle of length 4 would have to come from the relation $a^{-1}b^{-1}ab = e$; but $a^{-1}b^{-1}ab \neq e$ (because Γ is not commutative), so we conclude there is no cycle of length 4. Consider the possibility of a cycle of length 6. Without loss of generality, we may assume a and a^{-1} each occur twice as arc-labels in the cycle, and b and b^{-1} each occur just once as arc-labels. Then it is easy to see that (allowing for interchange of a and a^{-1} and interchange of b and b^{-1}) a cycle of length 6 would have to come from the relation $a^{-1}a^{-1}b^{-1}aab = e$; this would imply $a^2 \in Z(\Gamma)$. Since $Z(\Gamma) = \langle x^{2^{m-k}}, y^{2^{m-k}} \rangle$ and because $m - k > 1$, this would imply $\overline{a}^2 \in \overline{\Gamma}^4$; hence $\overline{a} \in \overline{\Gamma}^2$. This would imply $a \in \Phi(\Gamma)$, a contradiction. ∎

(4.2) Corollary. *Let Γ be any of the groups described in Theorem 4.1, and let Λ be any metacyclic group of odd order, such that no irredundant generating set for Λ contains an element of order 3 (e.g., this condition is satisfied if $|\Lambda|$ is not divisible by 3). Then $\Gamma \times \Lambda$ is a metacyclic group satisfying the hypotheses of Theorem 3.3. Therefore, $\tilde{\gamma}(\Gamma \times \Lambda) = 2 + |\Gamma \times \Lambda|/2$.*

Proof. Because Γ is a 2-group and Λ has odd order, Lemma 2.7 implies that $\Gamma \times \Lambda$ is metacyclic.

a) The elements of order 2 in $\Gamma \times \Lambda$ are precisely the elements of order 2 in Γ, and the elements of order 3 in $\Gamma \times \Lambda$ are precisely the elements of order 3 in Λ. Hence, Lemma 2.4 implies

$$\Phi(\Gamma \times \Lambda) \supset \Phi(\Gamma) \cup \Phi(\Lambda) \supset \{\text{elements of order 2 or 3}\}.$$

b) Any 4-valent Cayley graph on $\Gamma \times \Lambda$ has some 4-valent Cayley graph on Γ as a homomorphic image. Since 4-valent Cayley graphs on Γ are known to be bipartite and have girth ≥ 8, it follows that every 4-valent Cayley graph on $\Gamma \times \Lambda$ is bipartite and has girth ≥ 8. ∎

5. Lower bound on the orientable genus

(5.1) Proposition. *Suppose $G = \mathrm{Cay}(\Gamma; x^{\pm 1}, y^{\pm 1})$ is a 4-valent Cayley graph of a finite group Γ, such that $\langle x \rangle \lhd \Gamma$, and assume G has the following properties:*
a) *G is bipartite and has girth 8; and*
b) *every cycle of length 8 in G is of the form $xyxy^{-1}x^{-1}yx^{-1}y^{-1}$ (perhaps after taking the inverse and/or a cyclic permutation of the relator).*
Then $\gamma(G) \geq 1 + |\Gamma|/4 + n/20$, where $n = |\Gamma/\langle x \rangle|$.

Proof. Consider an orientable 2-cell embedding of G on some surface of genus γ. Let $\mathbb{O}$ be the collection of all the ordered pairs of the form (F, α), where F is a 2-cell of the embedding, α is an edge of F, and the face F is *not* an octagon.

Step 1. For any cycle C of the form x^m, the collection $\mathbb{O}$ contains at least two pairs (F, α) in which α belongs to C. First of all, suppose there is some vertex v of C at which the y-edge into v and the y-edge out of v both lie on the same side of C, as shown in Figure 2. Then the boundary of some face F contains both the x-edge into v and the x-edge out of v; thus the boundary of F has two consecutive x-edges. By assumption (b), we see that F is not an octagon. Because F contains two different edges of C, this yields the desired conclusion. We henceforth assume that, at each vertex of C, the y-edge into the vertex and the y-edge out of the vertex are on opposite sides of C.

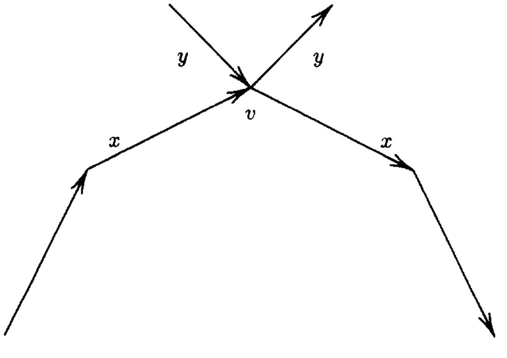

Fig. 2. A vertex at which both y-edges are on the same side of the x-cycle C

Consider x-edges α and β, as shown in Figure 3, on a typical octagon in the embedding. One can see that there is a y-edge to the left after α on the boundary of the octagon, and there is a y-edge to the right after β. (Alternatively, there could be a y^{-1}-edge to the left after α and a y^{-1}-edge to the right after β.) Because α and β are separated by yxy^{-1}, which is a power of x, we know that α and β lie on the same x-cycle, say C. Thus, we have seen that, as we traverse C, we will encounter a vertex v at which there is a y-edge leaving to the left, but at the next vertex, w, the y-edge leaves to the right, as shown in Figure 4. (Recall that the embedding is orientable.) Then the boundary of one of the faces containing the edge vw must contain the sequence $y^{-1}xy^{-1}$, and the boundary of the other face must contain the sequence yxy. On the other hand, in the relator $xyxy^{-1}x^{-1}yx^{-1}y^{-1}$ every occurrence of x is surrounded by y on one side and y^{-1} on the other. Hence, neither of the faces containing vw is an octagon, which yields the desired conclusion.

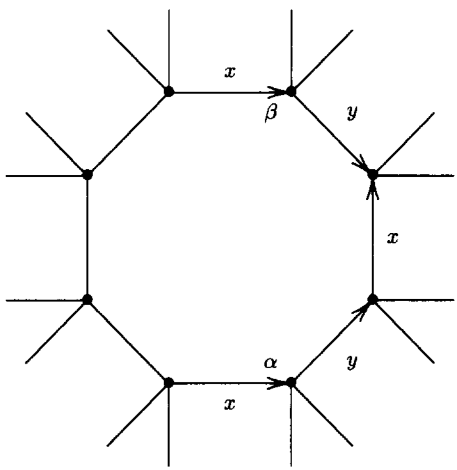

Fig. 3. A typical octagon

Step 2. We have $|\mathcal{O}| \geq 4n$. There are precisely n cycles of the form x^m, and these are all disjoint, so it follows from Step 1 that $\mathcal{O}$ contains at least $2n$ pairs (F, α) in which α is an x-edge. From assumption (b), we know that each octagon contains an equal number of x-edges and y-edges, so we conclude that $\mathcal{O}$ must also contain at least $2n$ pairs (F, α) in which α is a y-edge. Thus, as desired, $\mathcal{O}$ must contain at least $4n$ pairs all together.

Step 3. We have $\gamma \geq 1 + |\Gamma|/4 + n/20$. Because G is bipartite, any non-octagonal face must have at least 10 sides. For each k, let f_k be the number of k-gonal faces of the embedding. Then $10f_{10} + 12f_{12} + \ldots = |\mathcal{O}|$, so $f_{10} + f_{12} + \ldots \leq |\mathcal{O}|/10$. Because there are $2|\Gamma|$ edges in G, we must have

$$4|\Gamma| = 8 \cdot f_8 + 10 \cdot f_{10} + 12 \cdot f_{12} + \ldots = 8f_8 + |\mathcal{O}|,$$

so $f_8 = |\Gamma|/2 - |\mathcal{O}|/8$. Therefore

$$f_8 + (f_{10} + f_{12} + \ldots) \leq \left(\frac{|\Gamma|}{2} - \frac{|\mathcal{O}|}{8}\right) + \frac{|\mathcal{O}|}{10} = \frac{|\Gamma|}{2} - \frac{|\mathcal{O}|}{40}.$$

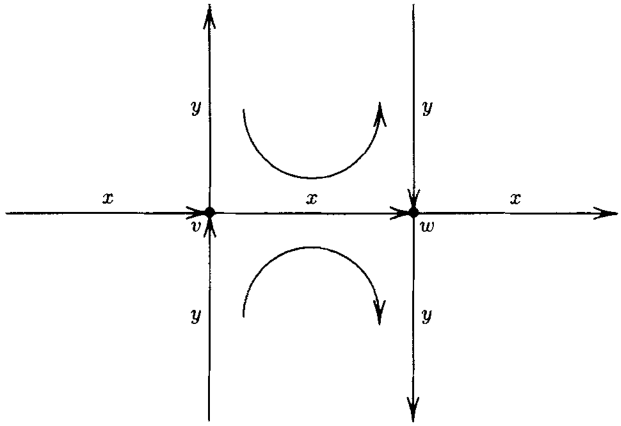

Fig. 4. A part of an x-cycle

By using the Euler formula together with the fact that G has exactly $|\Gamma|$ vertices and $2|\Gamma|$ edges, we have

$$\gamma = \frac{1}{2}(2 + |\Gamma| - (f_8 + f_{10} + f_{12} + \dots)) \geq \frac{1}{2}\left(2 + |\Gamma| - \left(\frac{|\Gamma|}{2} - \frac{|\mathcal{O}|}{40}\right)\right) = 1 + \frac{|\Gamma|}{4} + \frac{|\mathcal{O}|}{80}.$$

The desired conclusion now follows from Step 2. ∎

(5.2) Theorem. *Let m, n, and k be natural numbers, with $8 \leq 2k < m \leq n + k$, and let Γ be the metacyclic group given by the following generators and relations:*

$$\langle x, y \mid x^{2^m} = y^{2^n} = e,\ y^{-1}xy = x^{2^k+1} \rangle.$$

Then the Cayley graph $G = \operatorname{Cay}(\Gamma; x^{\pm 1}, y^{\pm 1})$ satisfies the hypotheses of Theorem 3.3 and the hypotheses of Proposition 5.1. Hence $2\gamma(G) - \tilde{\gamma}(G) \geq 2^n/10$.

Proof. (a) Theorem 4.1 asserts that Γ satisfies the hypotheses of Theorem 3.3. In particular, G is bipartite and has girth 8.

(b) To check that $xyxy^{-1}x^{-1}yx^{-1}y^{-1}$ is the only 8-relator, we can consider each possible 8-relator and reduce it to the canonical form $x^a y^b$ by using the relation $y^{-1}x = x^{2^k+1}y^{-1}$. If $a \not\equiv 0 \pmod{2^m}$ or $b \not\equiv 0 \pmod{2^n}$, then the word under consideration is not a relator. Note first that, by Lemma 2.6, we need only consider 8-relators in which x and x^{-1} occur the same number of times, and y and y^{-1} occur the same number of times. For convenience, let $q = 1 + 2^k$, so $y^{-1}x^r = x^{qr}y^{-1}$.

It is easy to see that (up to a cyclic permutation and inverses) the only possible 8-relator containing three y's (and hence also three y^{-1}'s) is $x^{-1}y^{-3}xy^3$. Because $x^{-1}y^{-3}xy^3 = x^{-1+q^3}$ and $-1 + q^3 = -1 + (1 + 2^k)^3 = 3 \cdot 2^k + 3 \cdot 2^{2k} + 2^{3k} \not\equiv 0 \pmod{2^m}$, we see that $x^{-1}y^{-3}xy^3$ is not a relator. Similarly, the only possible 8-relator containing three x's is $x^{-3}y^{-1}x^3y = x^{-3+3q}$. Because $-3 + 3q = 3 \cdot 2^k \not\equiv 0 \pmod{2^m}$, this is not a relator either.

Consider now an 8-relator containing two consecutive y's or two consecutive y^{-1}'s. We may assume the relator has two consecutive y's, by inverting if necessary. If the relator also has two consecutive y^{-1}'s, then the only possibility is $x^{-2}y^{-2}x^2y^2 = x^{-2+2q^2}$, but $-2+2q^2 = 2^{k+2}+2^{2k+1} \not\equiv 0 \pmod{2^m}$. So there cannot be two consecutive y^{-1}'s, which means the relator is of the form $x^ay^{-1}x^by^{-1}x^cy^2$, where $a+b+c=0$, $|a|+|b|+|c|=4$, and each of a,b,c is ±1 or ±2. We have $x^ay^{-1}x^by^{-1}x^cy^2 = x^{a+qb+q^2c}$. Because $q=1+2^k$ and $a+b+c=0$, we have $a+qb+q^2c = 2^k(b+2c)+2^{2k}c$. Because $2^k \ll 2^{2k} < 2^m$ and each of b and c is ±1 or ±2, it is not hard to see that $2^k(b+2c)+2^{2k}c$ cannot be congruent to 0 modulo 2^m. Hence we do not have a relator.

Consider, finally, an 8-relator with no consecutive y's and no consecutive y^{-1}'s. Then there are no consecutive x's or x^{-1}'s either; the relator must be of the form $x^ay^{-1}x^by^{-1}x^cyx^dy$ or $x^ay^{-1}x^byx^cy^{-1}x^dy$, where $a+b+c+d=0$ and each of a,b,c,d is ±1. (Either y^{-1} and y don't alternate, or they do.) We have $x^ay^{-1}x^by^{-1}x^cyx^dy = x^{a+q(b+d)+q^2c} = x^{2^k(b+d+2c)+2^{2k}c} \neq e$, so this is not a relator. We have $x^ay^{-1}x^byx^cy^{-1}x^dy = x^{a+c+q(b+d)} = x^{2^k(b+d)}$. If this is a relator, we must have $b=-d$, and then $a=-c$, because $a+b+c+d=0$. Thus any 8-relator must be of the form $x^ay^{-1}x^byx^{-a}y^{-1}x^{-b}y$, with $a,b=\pm1$. If $a=-b$, then a cyclic permutation of this relator is of the desired form $xyxy^{-1}x^{-1}yx^{-1}y^{-1}$; if $a=b$, then a cyclic permutation of the inverse is of the desired form. ∎

Example. Let Γ be any of the groups described in Theorem 5.2; let $u=xy$ and $v=y$. It is easy to see that u and v generate Γ, and that they satisfy the relation

$$u^2v^{-1}u^{-1}v^2u^{-1}v^{-1} = xyxyy^{-1}y^{-1}x^{-1}y^2y^{-1}x^{-1}y^{-1} = xyxy^{-1}x^{-1}yx^{-1}y^{-1} = e.$$

Thus, nonstandard generating sets for Γ can lead to the relators eliminated in part (b) of the proof of Theorem 5.2. We do not know whether it is possible to build an orientable, octagonal embedding for some other Cayley graph on Γ by using these other 8-relators. Hence, we are unable to extend Theorem 5.2 to an interesting lower bound on the orientable genus of the group Γ rather than just a lower bound on the orientable genus of the Cayley graph $\mathrm{Cay}(\Gamma; x^{\pm1}, y^{\pm1})$.

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